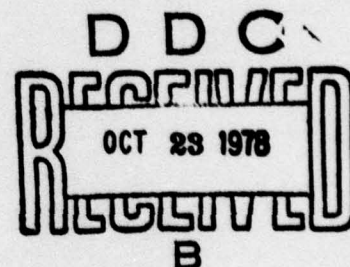(12) **LEVEL** II

EFFICIENCY IN INTEGRAL FACILITY DESIGN PROBLEMS

Research Report No. 78-11

by

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An example of a design might be a warehouse floor, represented by a set S, of area A, but unspecified shape. Given m warehouse users, we suppose user i has a known disutility function f_i such that $H_i(S)$, the integral of f_i over the set S (for example, a total travel distance), defines the disutility of the design S to user i. For the vector $H(S)$ with entries $H_i(S)$, we study the vector minimization problem over the set $\langle H(S): S$ a design$\rangle$, and call a design efficient if and only if it solves this problem. Assuming a mild regularity condition, we give necessary and sufficient conditions for a design →

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with $1 < p < \infty$ a design is efficient if and only if it is essentially the
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ABSTRACT

An example of a design might be a warehouse floor, represented by a set S, of area A, but unspecified shape. Given m warehouse users, we suppose user i has a known disutility function f_i such that $H_i(S)$, the integral of f_i over the set S (for example, a total travel distance), defines the disutility of the design S to user i. For the vector H(S) with entries $H_i(S)$, we study the vector minimization problem over the set {H(S): S a design}, and call a design efficient if and only if it solves this problem. Assuming a mild regularity condition, we give necessary and sufficient conditions for a design to be efficient, as well as verifiable conditions for the regularity condition to hold. For the case where f_i is the ℓ_p distance from warehouse dock i, with $1 < p < \infty$, a design is efficient if and only if it is essentially the same as a contour set of some Steiner-Weber function, $f_\lambda = \lambda_1 f_1 + \cdots + \lambda_m f_m$, when the λ_i are nonnegative constants, not all zero.

INTRODUCTION

We consider the problem of characterizing efficient designs. A "design" may be a planar set of known total area A, but unspecified shape. Any design must be contained in some given planar set L. As an example of a design, let S be a warehouse floor, and L be the lot of land in which the design will lie. Assume the design will have m users, with user i having a disutility function f_i, where $f_i(x)$ is the disutility of the point x in S to user i. For the warehouse problem $f_i(x)$ might be the distance (perhaps weighted by the frequency of use) which user i must travel, or pay to have traveled, in order to pick up an item stored at x.

For a given design S, define $H_i(S)$, for $1 \leq i \leq m$, to be the integral of the function f_i over the set S, so that $H_i(S)$ represents the total disutility of the design S to customer i, while $H(S) = (H_1(S), \ldots, H_m(S))^T$ represents the disutility vector of the design S for all users. We call a design S*
efficient if whenever any design S satisfies $H(S) \leq H(S*)$, then it must be true that $H(S) = H(S*)$. An efficient design thus solves a multiple objective optimization problem, and is Pareto optimal [10]. An evaluation of a design as given by $H_i(S)$ may occur when users are concerned about total disutility. In the warehouse context the disutility of S to user i, $H_i(S)$, might be appropriate when the users have to pay for total operating costs, and cannot agree upon a single-valued disutility function. The total operating costs for a particular user might be taken as proportional to the total travel distance due to storing this user's items in the warehouse. Alternatively, under an equal likelihood assumption, that is, a random storage policy in a warehouse (any item is equally likely to be stored at any location within the warehouse), each total operating cost, when divided by the constant A, the area of the warehouse, becomes an average operating cost. Under these

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circumstances, the problem under consideration is then one of minimizing the vector of average operating costs incurred by the users due to item movement within the warehouse. A third interpretation of the design problem is as a multiple objective location problem where the design S is to be located, S cannot be idealized as a point, and $H_i(S)$ is an average distance of S from "existing facility i."

We remark that in many cases efficient designs should best be viewed as design guidelines, rather than as final answers. Further, as is typically the case with multiple objective optimization problems, many different designs can be efficient, and the problem remains of choosing among such designs. Nevertheless we feel that the knowledge of efficient designs should be of value in helping to delimit the comparison of alternatives as well as to define more sharply the design problem(s) of interest.

To the best of our knowledge, the only research on the efficient design problem for the case m > 1 is by Chalmet, Francis, and Lawrence [3], where instead of the disutility of a design to user i being the integral of a disutility function f_i over the design, the supremum of the disutility function f_i over the design is used. They develop necessary and sufficient conditions for a design to be efficient, given very weak assumptions about the disutility functions, and draw additional conclusions when the functions are convex. For the case where m = 1, Francis [7] has studied a facility design problem; a _family_ of such problems is equivalent to the efficient design problem we consider. The establishment of this equivalence is one of our results. Corley and Roberts [4], as well as Lowe and Hurter [13] have considered somewhat similar design problems in regional partitioning and market area contexts. Lowe and Hurter employ the concept of "flat spots" of functions. This concept also arises in our analysis, although for different reasons.

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On considering a design to be a point, we
obtain the analogous problem of finding Pareto optimal
solutions for the vector $(f_1(x), \ldots, f_m(x))$. For this latter problem there
is, of course, a substantial literature ([10], [16], [18]), the discussion of
which is beyond the scope of this paper. We single out, however, the case
where each function f_i is a "planar" ℓ_p distance from an existing facility
at location q_i, say $f_i(x) = d_p(x, q_i)$, as this case is quite close in spirit
to the warehouse design problem, and has provided much of the impetus for the
study of the efficient design problem. For the case where the distances are
Euclidean, Kuhn [11] has demonstrated that the set of all Pareto optimal solu-
tions is just the convex hull of $q_1, \ldots, q_m$. Subsequently Wendell, Hurter,
and Lowe [17] have studied the problem for the case where the distances are
ℓ_p distances, and concentrated upon developing an algorithm for finding all
Pareto optimal solutions when $p = 1$. Their work has in turn motivated work by
Chalmet and Francis, who give a geometrical solution procedure [1], as well as
an order m log m algorithm [2] for the case $p = 1$.

We now give an overview of the paper. In Section 2, after introducing
some definitions and useful notation, we develop necessary conditions for a
design to be efficient. We show that if a design S* is efficient, then there
exists a nonzero column vector λ with nonnegative entries, such that $\lambda^T H(S*)$
$\leq \lambda^T H(S)$ for every design S. With M denoting the range of H, the set of all
vectors y such that $y = H(S)$ for some design S, we introduce in Section 3 a
condition, called the Support Regularity Condition (S.R.C.). We say that M
satisfies the S.R.C. whenever every supporting hyperplane P of M, where P =
$\{x \in E^m : \lambda^T x = \beta\}$, $\lambda \in E^m$, $0 \neq \lambda \geq 0$, intersects M in a single point. We
subsequently show that when M satisfies the S.R.C., the necessary conditions
for efficiency, developed in Section 2, are also sufficient. Combining these

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results, we obtain our main result of this paper in Section 3. Assuming the S.R.C. holds, a design $S*$ is efficient if and only if $S*$ satisfies the condition that there exists a nonzero column vector λ with nonnegative entries such that $\lambda^T H(S*) \leq \lambda^T H(S)$ for every design S. This condition can also be characterized (equivalently) by the Neyman-Pearson Lemma. Assuming the S.R.C. holds, and defining f_λ by $f_\lambda = \lambda_1 f_1 + \ldots + \lambda_m f_m$, we show that a design $S*$ is efficient if and only if there exists a vector λ, $0 \neq \lambda \geq 0$, and a constant k such that the following inequalities are true "almost everywhere" (that is, they are true except possibly on sets of area zero): $f_\lambda(x) \leq k$ for $x \in S*$, and $f_\lambda(x) \geq k$ for $x \notin S*$.

In Section 4, we identify sufficient conditions for the S.R.C. to hold. We prove that if L is a convex set, if each function f_i is convex, and if the set of all minimizing points of f_λ has area zero, for every λ such that $0 \neq \lambda \geq 0$, then the S.R.C. is satisfied, so that we can apply the results of Section 3 to problems having these properties. In particular, if each f_i is some ℓ_p distance with $1 < p < \infty$, then the S.R.C. holds. Further, f_λ is just the function occurring in the Steiner-Weber problem [8], [11], with the λ_i being the "weights." In this case a design $S*$ is efficient if and only if there exists a vector λ, $0 \neq \lambda \geq 0$, such that $S*$ is a contour set of the Steiner-Weber function f_λ almost everywhere.

In the analysis to follow we use Lebesgue measure theory. However, all of the insight needed can be obtained by thinking of a measurable function as an integrable function, and the measure of a set as the area (or volume, or hypervolume) of a set.

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1. ASSUMPTIONS AND DEFINITIONS

We assume we are given a nonempty set L which is a Lebesgue-measurable subset of E^n, with $\mu(L) < \infty$, where $\mu(\cdot)$ denotes the measure of a set. Further, we are given real-valued measurable disutility functions $f_1, \ldots, f_m$ which have domain L. For each i, $1 \leq i \leq m$ we assume that

$$\int_L |f_i(x)| \, dx < \infty.$$

Let A be a given constant with $0 < A < \mu(L)$, and define a <u>design</u> S to be a subset of L, of measure A. Let $\mathcal{D}$ be the collection of all designs. Given $S \in \mathcal{D}$, we define

$$H_i(S) = \int_S f_i(x) \, dx, \qquad 1 \leq i \leq m,$$

$$H(S) = (H_1(S), \ldots, H_m(S))^T.$$

We call $H_i(S)$ <u>the total disutility of</u> S <u>to</u> i, and H(S) <u>the total disutility vector of</u> S.

In this paper we consider the problem of characterizing efficient designs. A design $S^* \in \mathcal{D}$ is called <u>efficient</u> if whenever a design S satisfies $H(S) \leq H(S^*)$, then it must be true that $H(S) = H(S^*)$. It is useful to define $M = H(\mathcal{D})$ to be the range of H.

2. NECESSARY CONDITIONS FOR EFFICIENCY

We first establish M is convex and compact; the development follows that of Dantzig and Wald [5].

<u>Lemma 1</u>: M, the range of H, is convex and compact.

<u>Proof</u>: Define $f_{m+1} = 1$, and given any measurable subset T of L, let

$$\gamma(T) = \left(\int_T f_1, \ldots, \int_T f_m, \int_T f_{m+1} \right).$$

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As T varies over all measurable subsets of L, the corresponding points $\gamma(T)$ obtained constitute a set, say M'. That is, M' is the range of γ, when the domain of γ consists of all measurable subsets of L. A theorem due to Lyapunov [14] (see also Lindenstrauss [12]) establishes that M' is both convex and compact. Further, for any design S, we note that

$$\gamma(S) = (H(S)^T, A),$$

since

$$\int_S f_{m+1} = \mu(S) = A$$

and

$$(\int_S f_1, \ldots, \int_S f_m) = H(S)^T.$$

Denote by $\gamma(\mathcal{A})$ the image of $\mathcal{A}$ under γ, $\{\gamma(S) : S \in \mathcal{A}\}$, and denote by $\mathcal{H}'$ the hyperplane

$$\mathcal{H}' = \{Y = (y_1, \ldots, y_m, y_{m+1}) \in E^{m+1} : y_{m+1} = A\}.$$

We observe that $\gamma(\mathcal{A}) = M' \cap \mathcal{H}'$. Since $\mathcal{H}'$ is closed and convex, and M' is compact and convex, $\gamma(\mathcal{A})$ is compact and convex.

Now define the projection P from E^{m+1} into E^m as follows:

$$P(y_1, \ldots, y_m, y_{m+1}) = (y_1, \ldots, y_m)^T;$$

that is, P projects the first m entries of any vector in E^{m+1} into E^m. Because P is a projection, the image under P of any compact and convex set in E^{m+1} is a compact and convex set in E^m. Thus, in particular, $P(\gamma(\mathcal{A}))$ is compact and convex. Clearly, $M = P(\gamma(\mathcal{A}))$. $\square$

Lemma 2. Given any design S, there exists an efficient design S* such that $H(S*) \leq H(S)$.

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Proof: Given $S \in \mathcal{D}$, let $x = H(S)$, so that $x \in M$. Consider the following minimization problem:

$$\min \quad \lambda^T y$$

subject to $\quad y \in M, \quad y \leq x$

i.e. $\quad y \in K = \{u \in M : u \leq x\}$,

where $\lambda \in E^m$ is any vector having all positive entries. Since M is compact, K is compact (and nonempty also, as $x \in K$). The extreme value theorem now implies there exists a vector y^*, $y^* \in K$, which solves the minimization problem. Now y^* is _efficient_, in the sense that there is no z in K such that $z \leq y^*$ and $z \neq y^*$; for if such a z exists then $\lambda^T z < \lambda^T y^*$, giving a contradiction. Since $y^* \in M$, there exists $S^* \in \mathcal{D}$ such that $y^* = H(S^*)$.

We claim that S^* is efficient. Indeed, if $\hat{S}$ is an arbitrary design in $\mathcal{D}$, then $z = H(\hat{S}) \in M$. If, moreover, $H(\hat{S}) \leq H(S^*)$, then $z \leq y^* \leq x$. Thus $z \in K$. As was seen above, $z \leq y^*$ implies $z = y^*$ and therefore $H(\hat{S}) = H(S^*)$, proving S^* to be efficient. $\square$

Let N be a closed convex set in E^m and let $F(x)$ be a convex vector mapping from N into E^m, that is, a function from N into E^m such that $F(\lambda x + (1-\lambda)y) \leq \lambda F(x) + (1-\lambda)F(y)$ for every x, y $\in$ N and every λ for which $0 < \lambda < 1$. A point $x°$ in N is _efficient_ if there exists no x in N such that $F(x) \leq F(x°)$ and $F(x) \neq F(x°)$.

The following statement is equivalent to Lemma 7.4.1 of Karlin [9].

Lemma 3. If x^* is an efficient point then there exists a vector $\lambda \in E^m$, with $\lambda \neq 0$, $\lambda \geq 0$, such that

$$\lambda^T F(x^*) \leq \lambda^T F(x)$$

for all x $\in$ N.

To apply this result to the efficient design problem, take $N = M = H(\mathcal{D})$, which we know to be convex (and compact as well). Take $F(x) = x$, clearly a convex mapping. Then x^*, $x \in M$ and $F(x) \leq F(x^*)$ implies $F(x) = F(x^*)$, so if S^*, $S \in \mathcal{D}$, $x^* = H(S^*)$, $x = H(S)$, and $H(S) \leq H(S^*)$ then $H(S) = H(S^*)$. Hence, we have the following necessary condition for efficiency.

Theorem 1. If $S^* \in \mathcal{D}$ and S^* is efficient, then there exists a vector $\lambda \in E^m$, $\lambda \neq 0$, $\lambda \geq 0$, such that

$$\lambda^T H(S^*) \leq \lambda^T H(S)$$

for all $S \in \mathcal{D}$.

3. SUFFICIENT CONDITIONS FOR EFFICIENCY

Theorem 1 gives a necessary condition for a design S* to be efficient. We will now show that this condition is also sufficient, provided the range M of H meets the following regularity condition.

Definition. We say that M = H($\mathcal{B}$) satisfies the Support Regularity Condition (S.R.C.) whenever every supporting hyperplane $\mathcal{N}$ of M,

$$\mathcal{N} = \{x \in E^m : \lambda^T x = \beta\} \quad ,$$

$\lambda \in E^m$, such that $0 \neq \lambda \geq 0$, intersects M in a single point.

Theorem 2. Assume M satisfies the S.R.C. If there exists $\lambda \in E^m$, $0 \neq \lambda \geq 0$, such that with S* $\in \mathcal{B}$,

$$\lambda^T H(S^*) \leq \lambda^T H(S) \tag{i}$$

for every S $\in \mathcal{B}$, then S* is an efficient design.

Proof. Let $\hat{S} \in \mathcal{B}$ be such that $H(\hat{S}) \leq H(S^*)$. We shall show $H(\hat{S}) = H(S^*)$. Since $H(\hat{S}) \leq H(S^*)$, the hypotheses for λ imply

$$\lambda^T H(\hat{S}) \leq \lambda^T H(S^*). \tag{ii}$$

Since (i) holds when S = $\hat{S}$, (i) and (ii) imply

$$\lambda^T H(\hat{S}) = \lambda^T H(S^*). \tag{iii}$$

Let $\beta = \lambda^T H(S^*)$ and $\mathcal{N} = \{x : \lambda^T x = \beta\}$. Then (iii) implies $H(S^*) \in \mathcal{N}$, so, with (i), we conclude $\mathcal{N}$ is a supporting hyperplane of M. Because $H(S^*) \in M$, $H(\hat{S}) \in M$, $\mathcal{N}$ intersects M at $H(S^*)$ and $H(\hat{S})$. Thus the S.R.C. implies $H(S^*) = H(\hat{S})$. Hence, S* is efficient. $\square$

We remark (in the spirit of the proof of Lemma 2 above) that if $\mathcal{N} = \{x : \lambda^T x = \beta\}$ is a supporting hyperplane of M, with every entry in λ positive, such that S* in $\mathcal{B}$ satisfies $\lambda^T H(S^*) \leq \lambda^T H(S)$ for all S $\in \mathcal{B}$, then it is also true that S* is efficient; hence, there are some cases when a design S* arising in this way is efficient even if the S.R.C. does not hold.

4. NECESSARY AND SUFFICIENT CONDITIONS FOR EFFICIENCY

As a result of Theorems 1 and 2, we have

Theorem 3. Assume M satisfies the S.R.C. Then $S* \in \mathcal{A}$ is efficient if and only if there exists $\lambda \in E^m$, $0 \neq \lambda \geq 0$, such that

$$\lambda^T H(S*) \leq \lambda^T H(S) \quad , \quad \text{for all } S \in \mathcal{A}. \tag{1}$$

For $S \in \mathcal{A}$, we note, with $\lambda = (\lambda_1, \ldots, \lambda_m)^T$, that

$$\lambda^T H(S) = \sum_{i=1}^{m} \lambda_i H_i(S) = \int_S (\sum_{i=1}^{m} \lambda_i f_i) \equiv \int_S f_\lambda$$

where $\quad f_\lambda = \sum_{i=1}^{m} \lambda_i f_i.$

Thus, the condition (1) is equivalent to

$$\int_{S*} f_\lambda \leq \int_S f_\lambda \quad , \quad \text{for all } S \in \mathcal{A}. \tag{2}$$

For known λ, Francis [7] has pointed out that the problem (2) may be solved using the following lemma, a special case of a result of Dantzig and Wald [5], for the Neyman-Pearson Lemma:

Lemma 4. The condition (2) is equivalent to the following condition (3): there exists a number k such that

$$f_\lambda(x) \leq k \quad \text{for almost all} \quad x \in S*$$
$$\tag{3}$$
$$f_\lambda(x) \geq k \quad \text{for almost all} \quad x \notin S*.$$

(As is customary, for a condition to hold for "almost all" points in a set means that the collection of all points in the set for which the condition does not hold has measure zero. Likewise, for a condition to hold "almost everywhere", abbreviated as a.e., means that the collection of all points for which the condition does not hold has measure zero.)

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Theorem 3 and Lemma 3 together provide our main result:

Theorem 4. Assume M satisfies the S.R.C. Then $S* \in \mathcal{D}$ is efficient if and only if there exists $\lambda \in E^m$, with $0 \neq \lambda \geq 0$, and there exists k, such that

$$f_\lambda(x) \leq k \quad \text{for} \quad \text{almost all} \quad x \in S*$$

$$f_\lambda(x) \geq k \quad \text{for} \quad \text{almost all} \quad x \notin S*.$$

5. SUFFICIENT CONDITIONS FOR THE S.R.C. TO HOLD

To obtain additional insight, we develop sufficient conditions for the S.R.C. to hold, and show that the S.R.C. holds for a large class of functions $f_1, \ldots, f_m$. We first introduce some useful notation. Let $B \equiv \mu(L)$, $0 < B < \infty$. Let g be a real-valued measurable function defined on L. For every real number k, define the following sets:

$$a(k) = \{x \in L : g(x) < k\}$$

$$b(k) = \{x \in L : g(x) = k\}, \quad ab(k) = a(k) \cup b(k)$$

$$c(k) = \{x \in L : g(x) > k\}, \quad bc(k) = b(k) \cup c(k).$$

We call b(k) a contour line of g (of value k). We say that g has a flat spot if some contour line of g has positive measure.

Lemma 5. Suppose g has no flat spot. If S is a measurable subset of L such that

$$g(x) \leq k \quad \text{for} \quad \text{almost all} \quad x \in S$$

$$g(x) \geq k \quad \text{for} \quad \text{almost all} \quad x \in \bar{S} \equiv L \sim S,$$

then $S = ab(k)$ a.e.

Proof. The hypotheses for S imply $S \subset ab(k)$ a.e. and $\bar{S} \subset bc(k)$ a.e., so that

$$\mu(S) \leq \mu(ab(k)) \tag{i}$$

and

$$\mu(\bar{S}) \leq \mu(bc(k)).$$

Because $\mu(b(k)) = 0$,

$$\mu(\bar{S}) \leq \mu(c(k)). \tag{ii}$$

Hence

$$B = \mu(S) + \mu(\bar{S}) \leq \mu(ab(k)) + \mu(c(k)) = B, \tag{iii}$$

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and so (i), (ii), and (iii) give $\mu(S) = \mu(ab(k))$, $\mu(\bar{S}) = \mu(c(k))$. Since $S \subset ab(k)$ a.e and $\mu(S) = \mu(ab(k))$, $S = ab(k)$ a.e. □

<u>Lemma 6</u>. Suppose g has no flat spots. If S° and S' both minimize $\int_S g$ over all $S \in \mathcal{A}$, then $S^\circ = S'$ a.e.

<u>Proof</u>. Given the hypotheses, the converse of the Neyman-Pearson Lemma, Theorem 3.1 of Dantzig and Wald [5], implies there exist k°, k' such that

$$S^\circ \subset ab(k^\circ) \ \text{a.e.}, \qquad \bar{S}^\circ \subset bc(k^\circ) \ \text{a.e.},$$

$$S' \subset ab(k') \ \text{a.e.}, \qquad \bar{S}' \subset bc(k') \ \text{a.e..}$$

Lemma 5 implies $S^\circ = ab(k^\circ)$ a.e. and $S' = ab(k')$ a.e. Without loss of generality, assume $k^\circ \leq k'$, so that $ab(k^\circ) \subset ab(k')$. Since $\mu(ab(k^\circ)) = A = \mu(ab(k'))$, $ab(k^\circ) = ab(k')$ a.e., so $S^\circ = ab(k')$ a.e. Hence $S^\circ = S'$ a.e. □

<u>Lemma 7</u>. If, for every $\lambda \in E^m$, $0 \neq \lambda \geq 0$, f_λ has no flat spots, then the S.R.C. is satisfied.

<u>Proof</u>. Let $\mathcal{N} = \{x : \lambda^T x = \beta\}$ be any supporting hyperplane of M, with $0 \neq \lambda \geq 0$ and $\beta \leq \lambda^T y$ for all $y \in M$. Let $\mathcal{N}$ intersect M at u and v, so that $\lambda^T u = \beta = \lambda^T v$ and there exists S^*, $S' \in \mathcal{A}$, such that $u = H(S^*)$, $v = H(S')$. Given any $S \in \mathcal{A}$, there exists $y \in M$, such that $y = H(S)$, and so $\lambda^T H(S^*) = \beta = \lambda^T H(S') = \beta \leq \lambda^T H(S)$, and hence S^* and S' both minimize $\int_S f_\lambda$. Lemma 6 thus implies $S^* = S'$ a.e. and hence

$$\int_{S^*} f_i = \int_{S'} f_i , \qquad 1 \leq i \leq m,$$

which means $H(S^*) = H(S')$, that is, $u = v$. Hence $\mathcal{N}$ intersects M at a unique point. □

We now consider the case where L is a given <u>convex</u> set in E^m and g is a convex function defined on L. For any real number k, define the set

$$T_k = \{x \in L : g(x) \leq k\}.$$

T_k is convex. We denote the interior and boundary of T_k by $int(T_k)$ and ∂T_k respectively.

The following result is known and readily proven:

Lemma 8. If for at least one interior point y of T_k, $g(y) = k$, then $g(z) = k$ for every point z of T_k.

Corollary. If $g(z) < k$ for some $z \in T_k$, then $\{x \in L : g(x) = k\} \subset \partial T_k$.

Proof. By Lemma 8, if $g(z) < k$ for some z in T_k, then $g(y) < k$ for every interior point y of T_k. Hence if $x \in T_k$ and $g(x) \geq k$ (equivalent to $g(x) = k$), x is not an interior point of T_k, and so is a boundary point of T_k. □

Lemma 9. If S is any convex (and measurable) set in E^m, the boundary of S has measure zero.

Proof. See Eggleston [6], p. 73.

Theorem 5. Assume L and g are convex (and Lebesgue measurable). If the collection of all minimizing points of g has measure zero, then g has no flat spots.

Proof. Let $\mathcal{J} = \{x \in L : g(x) = k\}$ be any contour line of g. Since $\mu(\phi) = 0$, $\mu(\mathcal{J}) = 0$ if $\mathcal{J} = \phi$, so assume $\mathcal{J} \neq \phi$.

Consider the case where there exists no $z \in T_k$ such that $g(z) < k$. In this case $T_k = \mathcal{J}$, so every point in $\mathcal{J}$ is a minimizing point of g, and the hypotheses for g then imply $\mu(T_k) = 0$, so again, trivially, $\mu(\mathcal{J}) = 0$.

In the remaining case there exists $z \in T_k$ such that $g(z) < k$, so the Corollary to Lemma 8 implies $\mathcal{J} \subset \partial T_k$. Since T_k is measurable and convex, ∂T_k has measure zero by Lemma 9, in turn implying $\mathcal{J}$ has measure zero. □

Corollary. If, for every $\lambda \in E^m$, with $0 \neq \lambda \geq 0$, f_λ is convex, and the set of all points minimizing f_λ has measure zero, then the S.R.C. is satisfied.

We remark that f_λ is (strictly) convex for all λ, $0 \neq \lambda \geq 0$ iff $f_1, \ldots, f_m$ are (strictly) convex. Thus $f_1, \ldots, f_m$ strictly convex implies f_λ has at most one minimizing point, implying in turn that the S.R.C. is satisfied.

For many problems of interest, we might have $f_i(x) = d_p(x, q_i)$, $1 \le i \le m$, where $d_p(y, z)$, $y, z \in E^n$, represents the ℓ_p-distance between y and z, that is,

$$d_p(y, z) = [\sum_{j=1}^{n} |y_j - z_j|^p]^{\frac{1}{p}} .$$

Kuhn [11] observed that for $p = 2$, in order to have alternative minima for

$f_\lambda = \sum_{i=1}^{m} \lambda_i d_p(x, q_i)$, with $\lambda = (\lambda_1, \ldots, \lambda_m)^T \ge 0$, and at least two λ_i positive, the points q_i ($1 \le i \le m$) must be collinear. Actually, this observation is valid for any p with $1 < p < \infty$ (but may be invalid for $p = 1$ or for $p = \infty$, $p = \infty$ being the Tchebyshev distance case). In fact, f_λ is strictly convex for any $1 < p < \infty$, provided that the points q_i ($1 \le i \le m$) for which $\lambda_i > 0$ are not collinear. When the points q_i for which $\lambda_i > 0$ are contained in a line Λ, f_λ is strictly convex except on Λ, is convex on Λ, and either has a unique minimum or is minimal over a line segment contained in Λ. Hence for $1 < p < \infty$, f_λ is convex and the set of all points minimizing f_λ has measure zero, and so the S.R.C. is satisfied. The foregoing result is for the case where at least two λ_i are positive. If only one λ_i is positive, f_λ clearly has a unique minimum, and is convex, so that again the S.R.C. is satisfied.

From the discussion of the previous paragraph, when the f_i are ℓ_p distances with $1 < p < \infty$, we can use Theorem 4 and Lemma 5 to conclude that $S^* \in \mathcal{A}$ is efficient if and only if there exists $\lambda \in E^m$ with $0 \ne \lambda \ge 0$, and there exists k, such that $S^* = \{x \in L : f_\lambda(x) \le k\}$ a.e., that is, there exists $\lambda \in E^m$ with $0 \ne \lambda \ge 0$ such that S^* is some contour set of f_λ almost everywhere.

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